Newton's Interpolation Methods

P. Sam Johnson

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Overview

One of the basic ideas in Mathematics is that of a function and most useful tool of numerical analysis is **interpolation**.

According to Thiele (a numerical analyst), "**Interpolation is the art of reading between the lines of the table.**"

Broadly speaking, interpolation is the problem of obtaining the value of a function for any given functional information about it.

Interpolation technique is used in various disciplines like economics, business, population studies, price determination etc. It is used to fill in the gaps in the statistical data for the sake of continuity of information.

Overview

The concept of interpolation is the selection of a function $p(x)$ from a given class of functions in such a way that the graph of

$$y = p(x)$$

passes through a finite set of given data points. The function $p(x)$ is known as the **interpolating function** or **smoothing function**.

If $p(x)$ is a polynomial, then it is called the **interpolating polynomial** and the process is called the **polynomial interpolation**.

Similarly, if $p(x)$ is a finite trigonometric series, **we have trigonometric interpolation. But we restrict the interpolating function** $p(x)$ **to being a polynomial.**

Overview

The study of interpolation is based on the calculus of finite differences.

Polynomial interpolation theory has a number of important uses. Its primary uses is to furnish some mathematical tools that are used in developing methods in the areas of approximation theory, numerical integration, and the numerical solution of differential equations.

We discuss Newtons forward/backward formulae (for equally spaced nodes) and error bounds in **two** lectures.

A census of the population of the India is taken every 10 years. The following table lists the population, in thousands of people, from 1951 to 2011.

Year	1951	1961	1971	1981	1991	2001	2011
Population (in thousands)	361,088	439,235	548,160	683,329	846,388	1,028,737	1,210,193

In reviewing these data, we might ask whether they could be used to provide a reasonable estimate of the population, say, in 1996, or even in the year 2014. Predictions of this type can be obtained by using a function that fits the given data.

This process is called **interpolation / extrapolation**.

Introduction

If y is a function of x, then the functional relation may be denoted by the equation

$$y = f(x).$$

The forms of $f(x)$ can, of course, be very diverse, but we consider $f(x)$ as a polynomial of the nth degree in x

$$y = a_0 + a_1 x + \cdots + a_n x^n \quad (a_n \neq 0).$$

We call x as the **independent variable** and y as the **dependent variable**. It is usual to call x as **argument** and y as function of the argument or **entry**.

Since the polynomials are relatively simple to deal with, **we interpolate to the data by polynomials.**

Weierstrass Approximation Theorem

If the value of x whose corresponding value y is to be estimated lies within the given range of x, then it is a problem of **interpolation**. On the other hand, if the value lies outside the range, then it is a problem of **extrapolation**.

Thus for the theory of interpolation, it is not esssential that the functional form of $f(x)$ be known. The only information needed is the values of the function given for some values of the argument.

In the method of interpolation, it is assumed that the function is capable of being expressed as a polynomial. This assumption is based on **Weierstrass approximation theorem**. That is, the existence of an interpolating polynomial is supported by the theorem.

Weierstrass Approximation Theorem

Given any function, defined and continuous on a closed and bounded interval, there exists a polynomial that is as "close" to the given function as desired. This result is expressed precisely in the following theorem.

Theorem 1 (Weierstrass Approximation Theorem).

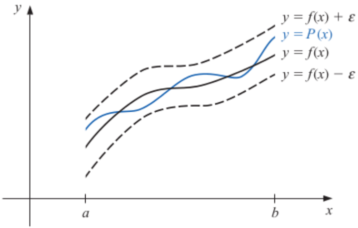
Suppose that f is defined and continuous on $[a, b]$. For each $\varepsilon > 0$, there exists a polynomial $p(x)$, with the property that

$$|f(x) - p(x)| < \varepsilon, \text{ for all } x \in [a, b].$$

Why polynomials are important?

Weierstrass approximation theorem is illustrated in the figure.

In science and engineering, polynomials arise everywhere.



An important reason for considering the class of polynomials in the approximation of functions is that the "derivative and indefinite integral of a polynomial" are easy to determine and they are also polynomials.

For these reasons, **polynomials are often used for approximating continuous functions.** We introduce various interpolating polynomials using the concepts of forward, backward and central differences.

Main Asssumption for Interpolation

There are no sudden jumps or falls in the values of the function from one period to another. This assumption refers to the smoothness of $f(x)$ i.e., the shape of the curve $y = f(x)$ changes gradually over the period under consideration.

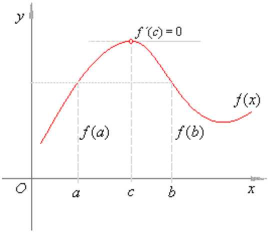
For example, if the population figures are given for, 1931, $1951, 1961, 1971$ and figures for 1941 are to be interpolated, we shall have to assume that the year 1941 **was not an exceptional year**, such as that affected by epidemics, war or other calamity or large scale immigration.

Rolle's theorem

We recall Rolle's theorem, which is useful in evaluating error bounds.

Theorem 2 (Rolle's Theorem).

Let f be continuous on $[a, b]$ and differentiable in (a, b). If $f(a) = f(b)$, then there is at least one point $c \in (a, b)$ such that $f'(c) = 0$.



Generalized Rolle's theorem

Theorem 3 (Generalized Rolle's Theorem).

Let f be continuous on $[a, b]$ and n times differentiable in (a, b). If $f(x)$ is zero at the $n + 1$ distinct numbers $c_0, c_1, \ldots, c_n$ in $[a, b]$, then a number c in (a, b) exists with $f^{(n)}(c) = 0$.

Error in Polynomial Interpolation

Let the function $y(x)$, defined by the $(n+1)$ points

$$(x_i, y_i), \qquad i = 0, 1, 2, \ldots, n$$

be continuous and differentiable $(n+1)$ times, and let $y(x)$ be approximated by a polynomial $p_n(x)$ of degree not exceeding n such that

$$p_n(x_i) = y_i$$

for $i = 0, 1, 2, \ldots, n$.

Error in Polynomial Interpolation

Using the polynomial $p_n(x)$ of degree n, we can obtain approximate values of $y(x)$ at some points other $x_i,\ 0 \leq i \leq n$.

Since the expression $y(x) - p_n(x)$ vanishes for $x = x_0, x_1, \ldots, x_n$ we put

$$y(x) - p_n(x) = L\pi_{n+1}(x) \tag{1}$$

where

$$\pi_{n+1}(x) = (x - x_0)(x - x_1)\cdots(x - x_n)$$

and L is to be determined such that the equation (1) holds for any intermediate value of $x' \in (x_0, x_n)$. Clearly

$$L = \frac{y(x') - p_n(x')}{\pi_{n+1}(x')}. \tag{2}$$

Error in Polynomial Interpolation

We construct a function $F(x)$ such that

$$F(x) = y(x) - p_n(x) - L\pi_{n+1}(x) \tag{3}$$

where L is given by the equation (2) above.
It is clear that

$$F(x_0) = F(x_1) = \cdots = F(x_n) = F(x') = 0$$

that is, $F(x)$ vanishes $(n+2)$ times in the interval $x_0 \leq x \leq x_n$.

Error in Polynomial Interpolation

Consequently, by the repeated application of Rolle's theorem, $F'(x)$ must vanish $(n+1)$ times, $F''(x)$ must vanish n times, etc,. in the interval $x_0 \leq x \leq x_n$. In particular, $F^{(n+1)}(x)$ must vanish once in the interval.

Let this point be given by $x = \xi, x_0 < \xi < x_n$. On differentiating the equation (3) $(n+1)$ times with respect to x and putting $x = \xi$, we obtain

$$F^{(n+1)}(\xi) = 0 = y^{(n+1)}(\xi) - L(n+1)!$$

so that

$$L = \frac{y^{(n+1)}(\xi)}{(n+1)!}. \tag{4}$$

Error in Polynomial Interpolation

Comparison of (2) and (4) yields the results

$$y(x') - p_n(x') = \frac{y^{(n+1)}(\xi)}{(n+1)!}\pi_{n+1}(x').$$

Dropping the prime on x', we obtain, for some $x_0 < \xi < x_n$,

$$y(x) - p_n(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_n)}{(n+1)!}y^{(n+1)}(\xi) \qquad (5)$$

which is the required expression for the error. Since $y(x)$ is, generally, unknown and hence we do not have any information concerning $y^{(n+1)}(x)$, formula (5) is **almost useless in practical computations.**

On the other hand, **it is extremely useful in theroetical work in different branches of numerical analysis.**

Newton's Interpolation Formulae for Equally Spaced Points

Given the set of $(n+1)$ values,

$$(x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n),$$

of x and y, it is required to find $p_n(x)$, a polynomial of the nth degree such that y and $p_n(x)$ agree at the tabulated points.

Let the values of x be equidistant,

$$x_i = x_0 + ih, i = 0, 1, 2, \ldots, n.$$

Since $p_n(x)$ is a polynomial of the nth degree, it may be written as

$$p_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0)(x - x_1)\cdots(x - x_{n-1}).$$

Polynomial Coefficients

Imposing the condition that y and $p_n(x)$ should agree at the set of tabulated points, we obtain

$$\begin{aligned}
a_0 &= y_0 \\
a_1 &= \frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta y_0}{h} \\
a_2 &= \frac{\Delta^2 y_0}{h^2 2!} \\
a_3 &= \frac{\Delta^3 y_0}{h^3 3!} \\
&\vdots \\
a_n &= \frac{\Delta^n y_0}{h^n n!}.
\end{aligned}$$

Interpolating Polynomial

Therefore

$$p_n(x) = y_0 + \frac{\Delta y_0}{h}(x - x_0) + \frac{\Delta^2 y_0}{h^2 2!}(x - x_0)(x - x_1) + \cdots$$

$$\cdots + \frac{\Delta^n y_0}{h^n n!}(x - x_0)(x - x_1)\cdots(x - x_{n-1})$$

is the polynomial of degree n agreeing with the (unknown) function y at the tabulated points.

Remainder Term (Error) in Polynomial Interpolation

Theorem 4.

Let $f(x)$ be a function defined in (a, b) and suppose that $f(x)$ have $n+1$ continuous derivatives on (a, b). If $a \leq x_0 < x_1 < \cdots < x_n \leq b$, then

$$f(x) - p_n(x) = \frac{(x - x_0)(x - x_1)\cdots(x - x_n)}{(n+1)!} f^{(n+1)}(\xi),$$

for some ξ between x and x_0 depending on $x_0, x_1, \ldots, x_n$ and f.

Newton's Forward Difference Interpolation Formula

Setting $x = x_0 + ph$ and substituting for $a_0, a_1, \ldots, a_n$, the above equation becomes

$$p_n(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots$$
$$\cdots + \frac{p(p-1)(p-2)\cdots(p-n+1)}{n!}\Delta^n y_0$$

which is **(Gregory)-Newton's forward difference interpolation formula** and is useful for interpolation **near the beginning** of a set of tabular values and is useful for extrapolating the values of y (to the left of y_0).

The first two terms of Newton's forward formula give the **linear interpolation** while the first three terms give a **parabolic interpolation** and so on.

Exercises

1. Prove that
   (a) $y_2 = y_0 + 2\Delta y_0 + \Delta^2 y_0$.
   (b) $y_3 = y_0 + 3\Delta y_0 + 3\Delta^2 y_0 + \Delta^3 y_0$.
2. Show that
   (a) $\Delta(\tan^{-1} x) = \tan^{-1}\left[\frac{h}{1+hx+x^2}\right]$.
   (b) $\Delta \log(1+4x) = \log\left[1 + \frac{4h}{1+4x}\right]$.
3. Evaluate
   (a) $\Delta^3(1-x)(1-2x)(1-3x)$ if $h = 1$.
   (b) $\Delta^{10}(1-x)(1-2x)(1-3x)\ldots(1-10x)$ taking $h = 1$.
   (c) $\Delta^{10}[(1-x)(1-2x^2)(1-3x^3)(1-4x^4)]$ if $h = 2$.
   (d) $\Delta^{10}[(1-ax)(1-bx^2)(1-cx^3)(1-dx^4)]$.

Exercises

1. If $f(x)$ and $g(x)$ are any two functions of x, prove that

$$\Delta[f(x)\ g(x)] = f(x)\ \Delta g(x) + g(x+h)\ \Delta f(x).$$

   Hence, evaluate $\Delta(x \ \sin \ x)$.

2. Evaluate
   (a) $\Delta^n(a^{bx+c})$.
   (b) $\Delta^n[\cos(ax+b)]$.

3. Prove that
   (a) $y_{n-2} = y_n - 2\nabla y_n + \nabla^2 y_n$.
   (b) $y_{n-3} = y_n - 3\nabla y_n + 3\nabla^2 y_n - \nabla^3 y_n$.

Exercises

1. Prove that
   (a) $(\Delta - \nabla)f(x) = \Delta\nabla f(x)$.
   (b) $\Delta\nabla$ and $\Delta - \nabla$ are equal operators. That is, $\Delta\nabla = \Delta - \nabla$.
2. Prove that $\Delta[f(x-1)\Delta g(x-1)] = \nabla[f(x)\Delta g(x)]$.
   [Hint : $\Delta[f(x)\ g(x)] = f(x)\ \Delta g(x) + g(x+h)\ \Delta f(x)$.]

Shift Operator

Let us recall the following expression for the first forward difference of a function $f(x)$ for equally spaced values of x with step-length h:

$$\Delta f(x) = f(x+h) - f(x).$$

This may be rewritten as

$$\begin{aligned} f(x+h) &= f(x) + \Delta f(x) \\ &= (1+\Delta) f(x). \end{aligned}$$

We denote $(1+\Delta)$ by E and call it as the (first order) shift operator. Thus,

$$\begin{aligned} E &= 1 + \Delta \\ Ef(x) &= f(x+h). \end{aligned}$$

Since Δ is a linear operator, it automatically follows that E is also a linear operator.

We define $E^2 f(x), E^3 f(x), E^4 f(x), \ldots$ by

$$\begin{aligned} E^2 f(x) &= E\{Ef(x)\} = Ef(x+h) = f(x+2h) \\ E^3 f(x) &= E\{E^2 f(x)\} = Ef(x+2h) = f(x+3h) \\ E^4 f(x) &= E\{E^3 f(x)\} = Ef(x+3h) = f(x+4h) \end{aligned}$$

and so on. In general, $E^n f(x)$ is defined by

$$E^n f(x) = f(x+nh), \; n = 1, 2, 3, \ldots.$$

E^n is called the nth order shift order as it shifts the value of the function at x to the value at $x+nh$. We can write the formula in an alternative form:

$$E^n y_m = y_{m+n}.$$

Shift Operator

Exercise 5.

Let the function $y = f(x)$ *take the values* $y_0, y_1, \ldots, y_n$ *corresponding to the values* $x_0, x_0 + h, \ldots, x_0 + nh$ *of* x. *Suppose* $f(x)$ *is a polynomial of degree* n *and it is required to evaluate* $f(x)$ *for* $x = x_0 + ph$, *where* p *is a any real number. Derive Newton's forward difference interpolation formula, by using shift operator* E.

[*Hint :* $y_p = f(x) = f(x_0 + ph) = E^p f(x_0) = (1 + \Delta)^p y_0$.]

Exercises

1. Evaluate
   (a) $E(x^2 - 3x + 1)$.
   (b) $(E^2 + 1)(x^3 + 2x^2 - 5x)$.
   (c) $E^{-1}(x^5 - x^3 + 1)$.
   (d) $E^{-3}(x^7 + x^5 + x^3 + x + 1)$.
2. Evaluate
   (a) $E^3 e^x$ and then $E^n e^x$, where x varies by a constant interval h.
   (b) $(E - 1)(E - 2)x$.
   (c) $(E - 3)(E + 4)3^x$.
   (d) $(E + 2)(E - 1)(2^{x/h} + x)$.
3. Expand the following :
   (a) $(2E - 3)(5\Delta + 4)x^2$.
   (b) $\Delta^2 E^{-3} x^5$ and prove that $\Delta^3 y_3 = \Delta^3 y_6$.

Exercises

1. Evaluate $\frac{\Delta^2}{E}\sin(x+h)+\frac{\Delta^2\sin(x+h)}{E\ \sin(x+h)}$.
2. Estimate the missing term in the following table :

x	0	1	2	3	4
y=f(x)	1	2	6	?	51

3. If $u_0 = 5, u_1 = 11, u_2 = 22, u_3 = 40, u_5 = 140$, then find u_4, given that the general term is represented by a fourth-degree polynomial.
4. Find $\log_{10}^{7}$ and $\log_{10}^{11}$ from the data given $f(x) = \log_{10}^{x}$ for some values of x.

x	6	7	8	9	10	11	12
$f(x) = \log_{10}^{x}$	0.77815	?	0.90309	0.95424	1.0000	?	1.07918

[Hint : 5 values of x and the corresponding values of $y = f(x)$ are given, so $\Delta^5 f(x) = 0$, for all x. By taking x as x_0 and x_1, we get two expressions in $y_0, y_1, \ldots, y_6$.]

Forward Difference Table

The values inside the boxes of the following difference table are used in deriving the Newton's forward difference interpolation formula.

Value of x	Value of $y = f(x)$	First Difference $\Delta f(x)$	Second Difference $\Delta^2 f(x)$	Third Difference $\Delta^3 f(x)$	Fourth Difference $\Delta^4 f(x)$
x_0	$\boxed{y_0}$				
		$\boxed{\Delta y_0}$			
$x_0 + h$	y_1		$\boxed{\Delta^2 y_0}$		
		Δy_1		$\boxed{\Delta^3 y_0}$	
$x_0 + 2h$	y_2		$\Delta^2 y_1$		$\boxed{\Delta^4 y_0}$
		Δy_2		$\Delta^3 y_1$	
$x_0 + 3h$	y_3		$\Delta^2 y_2$		
		Δy_3			
$x_0 + 4h$	y_4				

Error in Newton's Forward Difference Interpolation Formula

To find the error committed in replacing the function $y(x)$ by means of the polynomial $p_n(x)$, we obtain

$$y(x) - p_n(x) = \frac{(x - x_0)(x - x_1) \ldots (x - x_n)}{(n+1)!} y^{(n+1)}(\xi)$$

for some $\xi \in (x_0, x_n)$.

The error in the Newton's forward difference interpolation formula is

$$y(x) - p_n(x) = \frac{p(p - 1(p - 2) \cdots (p - n)}{(n+1)!} h^{n+1} y^{(n+1)}(\xi)$$

for some $\xi \in (x_0, x_n)$, and $x = x_0 + ph$.

Error in Newton's Forward Difference Interpolation Formula

As remarked earlier we do not have any information concerning $y^{(n+1)}(x)$, and therefore the above formula is useless in practice.

Neverthless, if $y^{(n+1)}(x)$ does not vary too rapidly in the interval, a useful estimate of the derivative can be obtained in the following way. Expanding $y(x+h)$ by Taylor's series, we obtain

$$y(x+h) = y(x) + hy'(x) + \frac{h^2}{2!}y''(x) + \cdots .$$

Neglecting the terms containing h^2 and higher powers of h, this gives

$$y'(x) \approx \frac{y(x+h) - y(x)}{h} = \frac{\Delta y(x)}{h}$$

Error in Newton's Forward Difference Interpolation Formula

Writing $y'(x)$ as $Dy(x)$ where $D \equiv d/dx$, the **differentiation operator**, the above equation gives the operator relations

$$D \equiv \frac{1}{h}\Delta \text{ and so } D^{n+1} \equiv \frac{1}{h^{n+1}}\Delta^{n+1}.$$

We thus obtain

$$y^{(n+1)}(x) \approx \frac{1}{h^{n+1}}\Delta^{n+1}y(x).$$

Hence the equation can be written as (equally spaced nodes, $x = x_0 + ph$)

$$y(x) - p_n(x) = \frac{p(p-1)(p-2)\cdots(p-n)}{(n+1)!}\Delta^{n+1}y(\xi)$$

for some $\xi \in (x_0, x_n)$, which is suitable for computation.

Newton's Backward Difference Interpolation Formula

Suppose we assume $p_n(x)$ in the following form

$$p_n(x) = a_0 + a_1(x - x_n) + a_2(x - x_n)(x - x_{n-1}) + \cdots$$

$$\cdots + a_n(x - x_n)(x - x_{n-1}) \cdots (x - x_1)$$

and then impose the condition that y and $p_n(x)$ should agree at the tabulated points $x_n, x_{n-1}, \ldots, x_2, x_1, x_0$, we obtain (after some simplification)

$$p_n(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \cdots + \frac{p(p+1)\cdots(p+n-1)}{n!}\nabla^n y_n$$

where $p = (x - x_n)/h$.

This is **(Gregory)-Newton's backward difference interpolation formula** and it uses tabular values to the left of y_n. This formula is therefore useful for interpolation **near the end of** the tabular values and is useful for extrapolating values of y (to the right of y_n).

Backward Difference Table

The values inside the boxes of the following difference table are used in deriving the Newton's backward difference interpolation formula.

Value of x	Value of $y = f(x)$	First Difference $\nabla f(x)$	Second Difference $\nabla^2 f(x)$	Third Difference $\nabla^3 f(x)$	Fourth Difference $\nabla^4 f(x)$
x_0	y_0				
		∇y_1			
$x_0 + h$	y_1		$\nabla^2 y_2$		
		∇y_2		$\boxed{\nabla^3 y_3}$	
$x_0 + 2h$	y_2		$\nabla^2 y_3$		$\boxed{\nabla^4 y_4}$
		∇y_3		$\boxed{\nabla^3 y_4}$	
$x_0 + 3h$	y_3		$\boxed{\nabla^2 y_4}$		
		$\boxed{\nabla y_4}$			
$x_0 + 4h$	$\boxed{y_4}$				

Error in Newton's Backward Difference Interpolation Formula

It can be shown that the error in this formula may be written as

$$y(x) - p_n(x) = \frac{p(p+1)(p+2)\cdots(p+n)}{(n+1)!} h^{n+1} y^{(n+!)}(\xi)$$

where $x_0 < \xi < x_n$ and $x = x_n + ph$.

Taylor's Theorem

Theorem 6.

Let $f(x)$ have $n+1$ continuous derivatives on $[a, b]$ for some $n \geq 0$, and let $x, x_0 \in [a, b]$. Then $f(x) = p_n(x) + R_n(x)$ where

$$p_n(x) = \sum_{k=0}^{n} \frac{(x - x_0)^k}{k!} f^{(k)}(x_0) \qquad \textbf{(}n\textbf{-degree polynomial)}$$

and

$$R_n(x) = \frac{(x - x_0)^{n+1}}{(n+1)!} f^{(n+1)}(\xi) \qquad \textbf{(error term)}$$

for some ξ between x and x_0.

Hence "ξ between x and x_0" means that either $x_0 < \xi < x$ or $x < \xi < x_0$ depending on the particular values of x and x_0 involved.

Taylor Polynomials

Here $p_n(X)$ is called the n**th Taylor polynomial** for f about x_0, and R_n is called the **remainder term** (or **truncation error**) associated with $p_n(x)$. Since the number ξ in the truncation error R_n depends on the value of x at which the polynomial $p_n(x)$ is being evaluated, it is a function of the variable x.

Taylor's theorem simply ensures that such a function exists, and that its value lies between x and x_0.

In fact, one of the common problems in numerical methods is to try to determine a realistic bound for the value of $f^{(n+1)}(\xi)$ when x is within some specified interval.

Taylor polynomials are not useful for interpolation

The Taylor polynomials are one of the fundamental building blocks of numerical analysis.

The Taylor polynomials agree as closely as possible with a given function at a specific point, but they concentrate their accuracy near that point.

A good interpolation needs to provide a relatively accurate approximation over an entire interval, and Taylor polynomials do not generally do this.

Example 7.

Taylor polynomials of various degree for $f(x) = 1/x$ *about* $x_0 = 1$ *are*

$$p_n(x) = \sum_{k=0}^{n} (-1)^k (x-1)^k.$$

When we approximate $f(3) = 1/3$ *by* $p_n(3)$ *for larger values of* n*, the approximations become increasingly inaccurate, as shown in the following table.*

n	0	1	2	3	4	5	6	7
$p_n(3)$	1	-1	3	-5	11	-21	43	-85

Taylor polynomials are not appropriate for interpolation

Since the Taylor polynomials have the property that all the information used in the approximation is concentrated at the single point x_0, it is not uncommon for these polynomials to give inaccurate approximations as we move away from x_0. This limits Taylor polynomial approximation to the situation in which approximations are needed only at points close to x_0.

For ordinary computational purposes it is more efficient to use methods that include information at various points.

The primary use of Taylor polynomials in numerical analysis is not for approximation purposes, but for the derivation of numerical techniques and for error estimation.

Since the Taylor polynomials are not appropriate for interpolation, alternative methods are needed.

The problem of determining a polynomial of degree one that passes through the distinct points (x_0, y_0) and (x_1, y_1) is the same as approximating a function f for which $f(x_0) = y_0$ and $f(x_1) = y_1$ by means of a first-degree polynomial interpolating, or agreeing with, the values of f at the given points.

We first define the functions

$$L_0(x) = \frac{x - x_1}{x_0 - x_1} \text{ and } L_1(x) = \frac{x - x_0}{x_1 - x_0},$$

and then define

$$p_1(x) = L_0(x)y_0 + L_1(x)y_1 = \frac{x - x_1}{x_0 - x_1}y_0 + \frac{x - x_0}{x_1 - x_0}y_1.$$

Since $L_0(x_0) = 1, L_0(x_1) = 0, L_1(x_0) = 0,$ and $L_1(x_1) = 1$, we have $p_1(x_0) = y_0$ and $p_1(x_1) = y_1$. So p_1 is the unique linear function passing (x_0, y_0) and (x_1, y_1).

Exercises 8.

1. *The table gives the distance in nautical miles of the visible horizon for the given heights (in feet) above the earth's surface.*

x	100	150	200	250	300	350	400
$y = f(x)$	10.63	13.03	15.04	16.81	18.42	19.90	21.27

Find the values of y when $x = 160$ and $x = 410$.

2. *From the following table, estimate the number of students who obtained marks between 40 and 45.*

Marks	30-40	40-50	50-60	60-70	70-80
No. of Students	31	42	51	35	31

3. *Find the cubic polynomial which takes the following values.*

x	0	1	2	3
$y = f(x)$	1	2	1	10

Also compute $f(4)$.

Exercises 9.

4. *In the table below, the values of y are consecutive terms of a series of which 23.6 is the 6th term. Find the first and tenth terms of the series.*

x	3	4	5	6	7	8	9
$y = f(x)$	4.8	8.4	14.5	23.6	36.2	52.8	73.9

5. *Using Newton's forward interpolation formula, show that*

$$\sum_{k=1}^{n} k^3 = \left\{ \frac{n(n+1)}{2} \right\}^2 .$$

An Observation of Central Differences

We derived Newton's forward and backward interpolation formulae which are applicable for interpolation near the beginning and end of tabulated values.

The following formulae are based on central differences which are best suited for interpolation near the middle of the table.

- Gauss's forward interpolation formula
- Gauss's backward interpolation formula
- Stirling's formula
- Bessel's formula
- Laplace-Everett's formula.

The coefficients in the above central difference formulae are smaller and converge faster than those in Newton's formulae.

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